

Fine singularity analysis of solutions to the Laplace equation: Berg's effect

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Abstract

We study Berg's effect on special domains. This effect is understood as monotonicity of a harmonic function (with respect to the distance from the center of a flat part of the boundary) restricted to the boundary. The harmonic function must satisfy piecewise constant Neumann boundary conditions. We show that Berg's effect is a rare and fragile phenomenon.

Keywords: singularities of harmonic functions, polygonal domains, piecewise constant Neumann data, Berg's effect

1 Introduction

We would like to apply the regularity results, we obtained in [11], to a study of the boundary behavior of harmonic functions. The motivation for this work comes from the observation made by Berg, [3], on crystals growing from a dilute solution. His conclusion may be expressed as follows, if the process is quasistatic, i.e. slow, and the crystal facets do not break nor bend, then the concentration, c , restricted to any facet, is an increasing function of the distance from the center of the facet. Since the process is quasistatic, then the concentration is a harmonic function outside of the crystal and the steady growth of facets implies that its normal derivative is constant on each facet.

A weaker version of Berg's effect is well-known in the physics literature (see e.g. [12, §3 eq.(6)]): the concentration at a facet center is smallest and its value at the edge of the facet is the largest.

There have been a few attempts to establish this rigorously, e.g. [13], the most recent one is [6]. The argument in [13] is based on explicit formulas. The authors of [6] tried to establish Berg's effect analyzing the boundary behavior of harmonic functions. However, the proof of regularity, [6, Lemma 1.] as noted in [11] contains a flaw. Thus the issue reopens. We will not resolve the problem studied in [6], but address a simpler one. Namely, we consider a planar domain Ω , defined as follows, for positive numbers r_1, r_2 and $\lambda_0 > 1$ we set

$$R_1 = (-r_1, r_1) \times (-r_2, r_2), \quad R_2 = \lambda_0 R_1, \quad \Omega = R_2 \setminus \overline{R_1}. \quad (1)$$

We define $\Gamma = \partial R_1$, $\tilde{\Gamma} = \partial R_2$. We consider the following problem,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \tilde{\Gamma}, \\ \frac{\partial u}{\partial \mathbf{n}} = u_n & \text{on } \Gamma. \end{cases} \quad (2)$$

As we noted, Neumann boundary condition is piecewise constant, i.e.

$$u_n = \begin{cases} a & \text{for } |x_2| = r_2, \\ b & \text{for } |x_1| = r_1. \end{cases} \quad (3)$$

where a and b are given numbers.

We have shown in [11] that for any pair $r_1, r_2 > 0$ defining Ω , there is a pair (a, b) (hence any other pair $(\lambda a, \lambda b)$, where $\lambda \neq 0$ will do) such that the unique weak solution to (2) with data (3), u , is in $C^1(\bar{\Omega})$.

In the present setting studying Berg's effect amounts to investigating the positivity of derivatives with respect to x or y of a weak solution to (2) with data (3), on the appropriate parts of the boundary of the inner rectangle.

Definition 1.1. *We assume that $a, b > 0$. We shall say Berg's effect holds for (Ω, a, b) if for u a weak solution of (2), (3) with boundary conditions u_n , it is true that*

$$xu_x \leq 0 \quad \text{on } \Gamma_1, \quad yu_y \leq 0 \quad \text{on } \Gamma_4. \quad (4)$$

Here Γ_1 and Γ_4 are perpendicular sides of the inner rectangle (see next section for notation). We show (see Theorem 3.1 for the proof):

Theorem 1.1. *Berg's effect holds for (Ω, a, b) if and only if u , a weak solution to (2) with data (3) is regular, i.e. $u \in C^1(\bar{\Omega})$.*

The C^1 -regularity of weak solutions is rather an exception not a rule, thus Berg's effect is rare. It is so, because it holds only for certain data a and b . We may say that the effect is not stable with respect to the perturbation of boundary data.

Theorem 1.2. *Assume that $r_1 = r_2$. Then Berg's effect holds for (Ω, a, a) , where $a > 0$. More generally, for any $r_1, r_2 > 0$ and $a > 0$ there exists a unique positive number b such that Berg's effect for (Ω, a, b) holds.*

In addition, perturbations of the domain destroy the effect. We can express it as follows. For $r_1, r_2 > 0$ and for $\varepsilon > 0$, we set

$$R_{1,\varepsilon} = (-r_1 - \varepsilon, r_1 + \varepsilon) \times (-r_2, r_2), \quad R_{2,\varepsilon} = \lambda_0 R_{1,\varepsilon}, \quad \Omega_\varepsilon = R_{2,\varepsilon} \setminus \overline{R_{1,\varepsilon}}, \quad (5)$$

Then, Ω_ε is a perturbation of $\Omega_0 = \Omega$ and we have

Theorem 1.3. *We assume that a is a given positive number. We take b is a unique positive number such that Berg's effect holds for (Ω, a, b) . Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ Berg's effect fails for $(\Omega_\varepsilon, a, b)$, where Ω_ε is defined above.*

There is also another question left open. Namely, what is the relation between a/b (or b/a) and the proportions of the rectangle r_2/r_1 . The current analysis does not give any clue besides the obvious statement: if $r_1 = r_2$, then a must be equal to b . We must write that despite its simplicity the problem has not been treated in the literature and the available studies of singularities of solutions to elliptic problems do not permit to make the desirable conclusion (at least in an obvious way). Possibly, tools used in the monographs [5], [8], [9], [10] were too general, while the methods used for numerical studies [1], [2], [4] were not able to capture the phenomenon we talk about.

Notation

Now, we recall the notation introduced in [11]. We shall write,

$$\begin{aligned} \Gamma &= \partial R_1 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup S_1 \cup S_2 \cup S_3 \cup S_4, \\ \tilde{\Gamma} &= \partial R_2 = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_3 \cup \tilde{\Gamma}_4 \cup \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3 \cup \tilde{S}_4, \end{aligned}$$

where $\Gamma_i, \tilde{\Gamma}_i$ are sides of rectangles and S_i, \tilde{S}_i are their vertexes, $i = 1, \dots, 4$. To be precise, we set $\Gamma_1 = \{(t, r_2) : t \in (-r_1, r_1)\}$, (resp. $\tilde{\Gamma}_1 = \{(t, r_1) : t \in (-r_2, r_2)\}$). The remaining sides of R_1 (resp. R_2), i.e. $\Gamma_2, \Gamma_3, \Gamma_4$, (resp. $\tilde{\Gamma}_j, j = 2, 3, 4$) are visited counterclockwise. We also set $S_i = \overline{\Gamma_i} \cap \overline{\Gamma_{i+1}}$, with the understanding that $\Gamma_{4+1} = \Gamma_1$ and we define \tilde{S}_j in the same manner. The distance from vertex S_i is denoted by ϱ_i .

For $i = 2, 4$, we set θ_i to be the angle measured at S_i from Γ_i to Γ_{i+1} . At the same time for $i = 1, 3$, we set θ_i to be the angle measured from Γ_{i+1} to Γ_i . Furthermore, by $\overline{\overline{S}}$ we denote the dual singular solution for Ω given by definition 2.1 [11].

2 Dual singular solutions

We proved in [11] that there are five possible forms of the zero level set of $\overline{\overline{S}}$. In this section, we shall show that in fact only one of them is attained. For this purpose we introduce an additional notation. For $\varepsilon > 0$, we set

$$R_{1,\varepsilon} = (-r_1 - \varepsilon, r_1 + \varepsilon) \times (-r_2, r_2), \quad R_{2,\varepsilon} = \lambda_0 R_{1,\varepsilon}, \quad \Omega_\varepsilon = R_{2,\varepsilon} \setminus \overline{R_{1,\varepsilon}}.$$

Let $\overline{\overline{S}} (\overline{\overline{S}}_\varepsilon$ resp.) be a dual singular solution constructed in definition 2.1 [11] for the domain Ω (Ω_ε resp.). First, we shall show the continuity of dual singular solutions with respect to small domain perturbations. More precisely, we have:

Lemma 2.1. *Let \overline{S} and \overline{S}_ε be as above. Then, for any K compact subset of $\overline{\Omega}$ which does not contain vertices $\{S_1, S_2, S_3, S_4\}$, the function \overline{S}_ε converges uniformly to \overline{S} on K .*

For the set Ω_ε , we adopt the same notation as for Ω , i.e. $\partial\Omega_\varepsilon = \Gamma_\varepsilon \cup \tilde{\Gamma}_\varepsilon$, the vertices of Γ_ε are $S_{i,\varepsilon}$, the vertices of $\tilde{\Gamma}_\varepsilon$ are denoted by $\tilde{S}_{i,\varepsilon}$, the sides of Γ_ε are $\Gamma_{i,\varepsilon}$ and the sides of $\tilde{\Gamma}_\varepsilon$ are $\tilde{\Gamma}_{i,\varepsilon}$, $i = 1, 2, 3, 4$.

Proof. We denote $U_\varepsilon = \Omega \cap \Omega_\varepsilon$. Thus, the boundary of U_ε consists of two parts: Γ_ε and $\tilde{\Gamma}$. On U_ε , we set

$$Q_\varepsilon(x, y) = \begin{cases} \overline{S}(x - \varepsilon \operatorname{sgn} x, y) & \text{for } |x| > \varepsilon, \\ \overline{S}(0, y) & \text{for } |x| \leq \varepsilon. \end{cases}$$

Then Q_ε is continuous on $x = \varepsilon$ and ∇Q_ε is also continuous on $x = \varepsilon$, because $\overline{S}_x(0, y) = 0$ by symmetry with respect to y -axis. Next, $\frac{\partial Q_\varepsilon}{\partial n} = 0$ on Γ_ε and $Q_\varepsilon = 0$ on $\tilde{\Gamma}_{1,\varepsilon}$ and $\tilde{\Gamma}_{3,\varepsilon}$. Further, \overline{S} is harmonic in Ω , bounded near $\tilde{\Gamma}$ and $\overline{S} = 0$ on $\tilde{\Gamma}$, hence

$$Q_\varepsilon = O(\varepsilon) \quad \text{on } \tilde{\Gamma}_2 \text{ and } \tilde{\Gamma}_4. \quad (6)$$

Furthermore, we shall show that for a constant c_0 , the estimate

$$|\Delta Q_\varepsilon| \leq c_0 \chi_{\{|x| < \varepsilon\}} \quad \text{on } U_\varepsilon, \quad (7)$$

holds, independently on ε , provided ε is small enough. Indeed, Q_ε is harmonic on the set $\{|x| > \varepsilon\}$ and $\Delta Q_\varepsilon(x, y) = \overline{S}_{yy}(0, y)$ on $\{|x| < \varepsilon\}$. Since \overline{S} is harmonic and smooth away from vertices (Proposition 2.1 [11]), hence we can bound $\overline{S}_{yy}(0, y)$ (estimates near Γ and $\tilde{\Gamma}$ are obtained after applying an appropriate reflection with respect to the boundary).

Finally, from $\|\overline{S}\|_{L^2(\Omega)} = 1$, we can deduce that

$$\|Q_\varepsilon\|_{L^2(U_\varepsilon)} = 1 + O(\varepsilon). \quad (8)$$

Now, let us suppose that $w_\varepsilon \in H^1(U_\varepsilon)$ is a unique weak solution of the following problem,

$$\begin{cases} \Delta w_\varepsilon = \Delta Q_\varepsilon & \text{in } U_\varepsilon, \\ \frac{\partial w_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_\varepsilon, \\ w_\varepsilon = Q_\varepsilon & \text{on } \tilde{\Gamma}, \end{cases}$$

Thus, from (6) and (7), we deduce that

$$\|w_\varepsilon\|_{L^2(U_\varepsilon)} = O(\varepsilon). \quad (9)$$

We set

$$v_\varepsilon = \frac{Q_\varepsilon - w_\varepsilon}{\|Q_\varepsilon - w_\varepsilon\|_{L^2(U_\varepsilon)}}. \quad (10)$$

Then, v_ε is a dual singular solution for domain U_ε , hence by Corollary 2.1 [11], v_ε satisfies (11)-(14) [11] for domain U_ε .

Let $w_{1,\varepsilon} \in H^1(U_\varepsilon)$ be a unique weak solution of the following problem

$$\begin{cases} \Delta w_{1,\varepsilon} = 0 & \text{in } U_\varepsilon \\ \frac{\partial w_{1,\varepsilon}}{\partial n} = 0 & \text{on } \Gamma_\varepsilon \\ w_{1,\varepsilon} = \bar{\bar{S}}_\varepsilon & \text{on } \tilde{\Gamma} \end{cases}$$

By definition, function $\bar{\bar{S}}_\varepsilon$ vanishes on $\tilde{\Gamma}_\varepsilon$, hence proceeding as earlier, we get $\bar{\bar{S}}_\varepsilon|_{\tilde{\Gamma}} = O(\varepsilon)$. Thus, standard estimate leads to

$$\|w_{1,\varepsilon}\|_{L^2(U_\varepsilon)} = O(\varepsilon). \quad (11)$$

We put

$$v_{1,\varepsilon} = \frac{\bar{\bar{S}}_\varepsilon - w_{1,\varepsilon}}{\|\bar{\bar{S}}_\varepsilon - w_{1,\varepsilon}\|_{L^2(U_\varepsilon)}}. \quad (12)$$

Then $v_{1,\varepsilon}$ satisfies (11)-(14) [11] for domain U_ε , therefore from Corollary 2.1 [11] we get $v_\varepsilon = \pm v_{1,\varepsilon}$. However, the singular parts of v_ε ($v_{1,\varepsilon}$ resp.) come from $\bar{\bar{S}}$ ($\bar{\bar{S}}_\varepsilon$ resp.), thus $v_\varepsilon = v_{1,\varepsilon}$, i.e.

$$Q_\varepsilon - w_\varepsilon = \frac{\|Q_\varepsilon - w_\varepsilon\|_{L^2(U_\varepsilon)}}{\|\bar{\bar{S}}_\varepsilon - w_{1,\varepsilon}\|_{L^2(U_\varepsilon)}}(\bar{\bar{S}}_\varepsilon - w_{1,\varepsilon}),$$

If we use (8), (9), (11) and $\|\bar{\bar{S}}_\varepsilon\|_{L^2(U_\varepsilon)} = 1 + O(\varepsilon)$, then

$$Q_\varepsilon - \bar{\bar{S}}_\varepsilon = w_\varepsilon - w_{1,\varepsilon} + O(\varepsilon)(\bar{\bar{S}}_\varepsilon - w_{1,\varepsilon}). \quad (13)$$

From conditions (9) and (11), by using the standard regularity argument, we obtain that for any fixed K , satisfying the Lemma assumption, w_ε and $w_{1,\varepsilon}$ converge uniformly to zero on K . Next, for K as above, $\bar{\bar{S}}_\varepsilon|_K$ is uniformly bounded with respect to ε , provided ε is small enough (see Proposition 2.1 [11]). Therefore,

$$Q_\varepsilon - \bar{\bar{S}}_\varepsilon \rightarrow 0 \quad \text{uniformly on } K. \quad (14)$$

Finally, we notice that $\bar{\bar{S}}$ is continuous on K , hence $Q_\varepsilon \rightarrow \bar{\bar{S}}$ uniformly on K . Therefore, the claim of the Lemma follows. \square

Remark 2.1. *The claim of Lemma 2.1 is also true if $\bar{\bar{S}}_\varepsilon$ is the dual singular solution defined for domain Ω_ε , when Ω_ε is a small perturbation of Ω in both directions, x -axis and y -axis. In this case we have to extend $\bar{\bar{S}}_\varepsilon$ across the boundary. It could be done by odd reflection with respect to $\tilde{\Gamma}_\varepsilon$ and by even reflection with respect to Γ_ε . The proof requires only minor modifications.*

Let us fix $\lambda_0 > 1$. Then, each $(r_1, r_2) \in \mathbb{R}_+^2$ defines domain $\Omega = \Omega(r_1, r_2)$ for which we construct a dual singular solution $\bar{\bar{S}} = \bar{\bar{S}}(r_1, r_2)$. In the proof of [11, Lemma 2.9], we indicate that zero level sets of $\bar{\bar{S}}$ may have five possible shapes. The k -th possibility is illustrated by k -th figure in [11], $k = 1, 2, 3, 4$ and the fifth possibility corresponds to the situation when the vertex S_1 and the side Γ_4 are

connected by a zero level set. We divide the set \mathbb{R}_+^2 into three parts, A_1 , A_2 , A_3 in the following way:

- $(r_1, r_2) \in A_1$ if for $\bar{\bar{S}}(r_1, r_2)$ the first or the forth possibility holds;
- $(r_1, r_2) \in A_2$ if for $\bar{\bar{S}}(r_1, r_2)$ the second or the fifth possibility holds;
- $(r_1, r_2) \in A_3$ if for $\bar{\bar{S}}(r_1, r_2)$ the third possibility holds. Then,

$$\mathbb{R}_+^2 = A_1 \cup A_2 \cup A_3. \quad (15)$$

In other words, $(r_1, r_2) \in A_1$ if a dual singular solution for $\Omega(r_1, r_2)$ is positive in a neighborhood of $\tilde{\Gamma}$, $(r_1, r_2) \in A_2$ if it is negative in a neighborhood of $\tilde{\Gamma}$ and $(r_1, r_2) \in A_3$ if $\bar{\bar{S}}$ changes sign in any neighborhood of $\tilde{\Gamma}$. We will show that Lemma 2.1 implies that the structure of zero level sets of $\bar{\bar{S}}$ remains unchanged under a small perturbation of the domain. More precisely,

Lemma 2.2. *The sets A_i are open.*

Proof. Let $(r_1, r_2) \in A_1$ and $\bar{\bar{S}}(r_1, r_2)$ be a corresponding solution for $\Omega = \Omega(r_1, r_2)$. For each $\varepsilon = (\varepsilon_1, \varepsilon_2)$, we have $\bar{\bar{S}}_\varepsilon$ defined for domain Ω_ε , where $\Omega_\varepsilon = R_{2,\varepsilon} \setminus \overline{R_{1,\varepsilon}}$, $R_{1,\varepsilon} = (-r_1 - \varepsilon_1, r_1 + \varepsilon_1) \times (-r_2 - \varepsilon_2, r_2 + \varepsilon_2)$, $R_{2,\varepsilon} = \lambda_0 R_{1,\varepsilon}$. We shall show that for ε_i close to zero the corresponding solution $\bar{\bar{S}}_\varepsilon$ has the zero level set as in the first or forth case. Firstly, we note that $\bar{\bar{S}}$ is positive in a neighborhood of $\tilde{\Gamma}$. We denote the set $R_2 \setminus (\lambda_0 - \delta)R_1$ by K , where δ is so small that $\bar{\bar{S}}_{|(\lambda_0 - \delta)\partial R_2} > 0$. For this set K , we use Lemma 2.1 (more precisely Remark 2.1). Then $\bar{\bar{S}}_\varepsilon \rightarrow \bar{\bar{S}}$ on K and hence $\bar{\bar{S}}_{\varepsilon|(\lambda_0 - \delta)\partial R_2} > 0$ for ε_i small enough. It means that for such $\varepsilon = (\varepsilon_1, \varepsilon_2)$, the first or the fourth possibility holds, i.e. $(r_1 + \varepsilon_1, r_2 + \varepsilon_2) \in A_1$, in other words set A_1 is open.

For set A_2 , we may proceed in a similar manner, because in this case the corresponding solution $\bar{\bar{S}}$ is negative in a neighborhood of $\tilde{\Gamma}$ and we can argue as earlier.

It remains to show that A_3 is open. First, we notice that if $(r_1, r_2) \in A_3$, then the corresponding solution $\bar{\bar{S}}$ has at least two points $x_1, x_2 \in \tilde{\Gamma}$, such that

$$\frac{\partial \bar{\bar{S}}}{\partial n}(x_1) > 0, \quad \frac{\partial \bar{\bar{S}}}{\partial n}(x_2) < 0. \quad (16)$$

Indeed, in this case the zero level set consists of four analytic curves, each of them connects vertex S_i with $\tilde{\Gamma}$. These curves divide Ω into four regions. In two of them $\bar{\bar{S}}$ is positive and in the other two regions $\bar{\bar{S}}$ is negative. Hence, in each of these regions, $\bar{\bar{S}}$ attains its supremum or infimum on $\tilde{\Gamma}$, because $\bar{\bar{S}}|_{\tilde{\Gamma}} = 0$. Then, by Hopf Lemma, we get (16).

Further, from Lemma 2.1 and Remark 2.1, we have uniform convergence of $\bar{\bar{S}}_\varepsilon$ to $\bar{\bar{S}}$ on compact subsets, which do not contain any vertex. But $\bar{\bar{S}}_\varepsilon$ are harmonic, thus their derivatives also converge uniformly to derivatives of $\bar{\bar{S}}$ on such compact subsets. Thus,

$$\frac{\partial \bar{\bar{S}}_\varepsilon}{\partial n}(x_1) > \frac{1}{2} \frac{\partial \bar{\bar{S}}}{\partial n}(x_1) > 0, \quad \frac{\partial \bar{\bar{S}}_\varepsilon}{\partial n}(x_2) < \frac{1}{2} \frac{\partial \bar{\bar{S}}}{\partial n}(x_2) < 0,$$

for ε small enough. Finally, we conclude that

$$\frac{\partial \bar{S}_\varepsilon}{\partial n}(x_{1,\varepsilon}) > 0, \quad \frac{\partial \bar{S}_\varepsilon}{\partial n}(x_{2,\varepsilon}) > 0,$$

for $x_{1,\varepsilon}, x_{2,\varepsilon} \in \tilde{\Gamma}_\varepsilon$, provided ε is sufficiently small.

Thus, the above inequalities imply that \bar{S}_ε is negative in a neighborhood of $x_{1,\varepsilon} \in \tilde{\Gamma}_\varepsilon$ and positive in a neighborhood of $x_{2,\varepsilon} \in \tilde{\Gamma}_\varepsilon$. This is so, because $\bar{S}_\varepsilon|_{\tilde{\Gamma}_\varepsilon} = 0$. By the continuity of \bar{S}_ε in Ω_ε , \bar{S}_ε attains zero on every curve connecting these neighborhoods. As a result, the zero level set of \bar{S}_ε connects Γ_ε and $\tilde{\Gamma}_\varepsilon$, provided that ε is sufficiently small. Hence, A_3 is open. \square

Theorem 2.1. *We assume that $r_1, r_2 > 0$, $\lambda_0 > 1$ and Ω is defined by (1). Let \bar{S} be the dual singular solution constructed in definition 2.1 [11] for the domain Ω . Then, W_0 , the zero level set of \bar{S} consists of four analytic curves. Each of them connects one vertex S_i with the outer part of the boundary $\tilde{\Gamma}$. In particular, $\bar{S}|_{\Gamma_1} < 0$ and $\bar{S}|_{\Gamma_2} > 0$, hence $\int_{\Gamma_1} \bar{S} < 0$ and $\int_{\Gamma_2} \bar{S} > 0$.*

Proof. From (15) and Lemma 2.2, we deduce that exactly one from the sets A_i is not empty. Lemma 2.10 [11] shows that A_3 is not empty, because $(r_1, r_1) \in A_3$. Thus $\mathbb{R}_+^2 = A_3$. It means that for each $(r_1, r_2) \in \mathbb{R}_+^2$, the structure of the level set is the same as in the third possibility, i.e. S_i and $\tilde{\Gamma}$ are connected by the zero level set. In order to prove the remaining statements, we note that \bar{S} is continuous on Γ_i and $\inf_{\Gamma_1} \bar{S} = -\infty$, $\sup_{\Gamma_2} \bar{S} = \infty$. If $\max_{\Gamma_1} \bar{S} = \bar{S}(x_0, r_2) \geq 0$, then \bar{S} restricted to the set $\{(x, y) \in \Omega : \bar{S}(x, y) < 0\}$ has a maximum on Γ_1 , thus by Hopf Lemma, the outer normal derivative at (x_0, r_2) is positive, which contradicts the definition of \bar{S} . A similar argument works for Γ_2 . \square

3 Berg's effect

Berg's effect is related to the boundary behavior of harmonic function, u , in a domain of the form $\Omega \setminus P$, where P is a polygon, and the normal derivative of u is constant on each side of P . Here, we study only harmonic functions, which are solutions to (2), (3).

Our main observation is that Berg's effect holds if and only if u is a regular solution to (2), (3).

Theorem 3.1. *Berg's effect holds for (Ω, a, b) if and only if the corresponding weak solution of (2), (3) is in $C^1(\bar{\Omega})$.*

Proof. Suppose that u is a weak solution of (2) and $u \notin C^1(\bar{\Omega})$. Then, by Corollary 2.2 [11] $a \int_{\Gamma_1} \bar{S} + b \int_{\Gamma_2} \bar{S} \neq 0$, hence there is a singular part of solution u and by Proposition 2.1 [11], we have

$$u = c_{2,1} \varrho_1^{\frac{2}{3}} \cos \frac{2}{3} \theta_1 + w \quad \text{on } B(S_1, \delta),$$

for a $\delta > 0$, where $w \in C^1(\overline{\Omega} \cap B(S_1, \delta))$ and $c_{2,1} \neq 0$ (see Section 1 for the definitions of ϱ_1 and θ_1). Due to the boundary conditions $w_x(S_1) < 0$ and $w_y(S_1) < 0$, hence by the continuity of ∇w , we get $w_x|_{\Gamma_1 \cap B(S_1, \delta_1)} < 0$ and $w_y|_{\Gamma_4 \cap B(S_1, \delta_1)} < 0$ for a $\delta_1 > 0$. On the other hand, $(\varrho_1^{\frac{2}{3}} \cos \frac{2}{3}\theta_1)_x|_{\Gamma_1} = -(r_1 - x)^{\frac{2}{3}}_x = \frac{2}{3}(r_1 - x)^{-\frac{1}{3}} \nearrow \infty$ if $x \rightarrow r_1^-$ and $(\varrho_1^{\frac{2}{3}} \cos \frac{2}{3}\theta_1)_y|_{\Gamma_4} = ((r_2 - y)^{\frac{2}{3}})_y = -\frac{2}{3}(r_1 - y)^{-\frac{1}{3}} \searrow -\infty$ if $y \rightarrow r_2^-$. Thus, for a $\delta_2 > 0$, we have $xu_x|_{\Gamma_1 \cap B(S_1, \delta_2)} > 0$ if $c_{2,1} > 0$ and $yu_y|_{\Gamma_4 \cap B(S_1, \delta_2)} > 0$ if $c_{2,1} < 0$. Thus, Berg's effect does not hold for (Ω, a, b) .

In order to prove the other implication, we suppose that u is a weak solution of (2), which belongs to $C^1(\overline{\Omega})$ but Berg's effect does not hold for (Ω, a, b) , i.e. $u_x(x, r_2) > 0$ for an $x \in (0, r_1)$ or $u_y(r_1, y) > 0$ for a $y \in (0, r_2)$. We shall show that the first possibility leads to a contradiction (the reasoning in the other case is the same). For this purpose we denote the set $\Omega \cap \{(x, y) : x > 0\}$ by Ω_+ and we shall consider u_x in Ω_+ . Then, u_x is harmonic in Ω_+ and continuous on $\overline{\Omega}_+$. First, we note that u is positive in Ω . Indeed, by definition $u|_{\tilde{\Gamma}} = 0$, thus if u were negative at a point of Ω , then u would have a negative minimum on Γ , but then we would get a contradiction with Hopf Lemma, because by definition $\frac{\partial u}{\partial n}|_{\Gamma} > 0$. Hence, u is positive in Ω and from boundary condition $u|_{\tilde{\Gamma}} = 0$, we deduce that $u_x|_{\tilde{\Gamma}_4} \leq 0$. Next, by condition $u|_{\tilde{\Gamma}} = 0$, we get $u_x|_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_3} = 0$ and by the symmetry of u with respect to $\{x = 0\}$, we obtain $u_x = 0$ on $\Omega \cap \{x = 0\}$. Finally, by definition (see (3)), $u_x|_{\Gamma_4} = -b$. Thus, if $u_x(x, r_2) > 0$ for an $x \in (0, r_1)$, then u_x restricted to Ω_+ admits a positive maximum, which is necessarily located on Γ_1 , say at $(x_0, r_1) \in \Gamma_1$. Then, by Hopf Lemma, we deduce that $\frac{\partial u_x}{\partial n}(x_0, r_1) > 0$, but on the other hand using boundary condition (3) we get $\frac{\partial u_x}{\partial n}(x_0, r_1) = \frac{\partial}{\partial x} \frac{\partial u}{\partial n}(x_0, r_1) = \frac{\partial}{\partial x} a = 0$, which yields a contradiction. Thus, u_x is non positive on the boundary of Ω_+ , hence it is negative in Ω_+ . In particular, $u_x|_{\Omega_+} \leq 0$, hence $xu_x|_{\Gamma_1} \leq 0$. The proof that $yu_y|_{\Gamma_4} \leq 0$ is analogous. \square

Proof of Theorem 1.2. If $r_1 = r_2$, then applying Theorem 1.2 [11], we get $u \in C^1(\overline{\Omega})$ and from Theorem 3.1 we deduce that Berg's effect holds for $(\Omega, 1, 1)$. In the general case, from Theorem 2.1 we deduce that numbers $\alpha_1 = \int_{\Gamma_1} \overline{S}$ and $\beta_1 = \int_{\Gamma_4} \overline{S}$ from Theorem 1.1 [11] are not zero, hence the claim follows from Theorem 1.1 [11] and Theorem 3.1. \square

In the above considerations, we analyze the stability of Berg's effect under a perturbation of boundary conditions. Below, we shall investigate its stability under a domain perturbation.

Proof of Theorem 1.3. Recall that domain Ω_ε is defined by (5) and positive number b is given by Corollary 1.2. Denote by u a weak solution of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = u_n & \text{on } \Gamma, \\ u = 0 & \text{on } \tilde{\Gamma}, \end{cases} \quad (17)$$

where u_n is defined in (3). Then, from Theorem 3.1 and [11, Proposition 2.1], we

have $u \in H^2(\Omega) \cap C^1(\overline{\Omega})$. We set,

$$u_\varepsilon(x, y) = \begin{cases} u(x - \varepsilon \operatorname{sgn} x, y) & \text{for } \varepsilon < |x| \leq r_1 + \varepsilon \\ u(0, y) & \text{for } |x| \leq \varepsilon \end{cases}$$

From the symmetry of u with respect to $\{x = 0\}$ we get $u_x(0, y) = 0$, hence $u_\varepsilon \in H^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega_\varepsilon})$ and

$$\Delta u_\varepsilon(x, y) = \begin{cases} 0 & \text{for } \varepsilon < |x| \leq r_1 + \varepsilon \\ u_{yy}(0, y) & \text{for } |x| \leq \varepsilon \end{cases}$$

We set $f_\varepsilon = \Delta u_\varepsilon$. Then, we notice that $u_\varepsilon = 0$ on $\tilde{\Gamma}_\varepsilon$ and $\frac{\partial u_\varepsilon}{\partial n} = u_{n,\varepsilon}$ on Γ_ε , where

$$u_{n,\varepsilon} = \begin{cases} a & \text{for } |x_2| = r_2, \\ b & \text{for } |x_1| = r_1 + \varepsilon. \end{cases} \quad (18)$$

Function f_ε belongs to $L^2(\Omega_\varepsilon)$. Hence, there exists a unique $w_\varepsilon \in H^1(\Omega_\varepsilon)$, a solution of the problem

$$\begin{cases} \Delta w_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ \frac{\partial w_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_\varepsilon, \\ w_\varepsilon = 0 & \text{on } \tilde{\Gamma}_\varepsilon. \end{cases} \quad (19)$$

We denote $v_\varepsilon = u_\varepsilon - w_\varepsilon$. Then, v_ε satisfies

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial v_\varepsilon}{\partial n} = u_{n,\varepsilon} & \text{on } \Gamma_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \tilde{\Gamma}_\varepsilon. \end{cases} \quad (20)$$

By Theorem 3.1, Berg's effect holds for $(\Omega_\varepsilon, a, b)$ if and only if v_ε is in $C^1(\overline{\Omega_\varepsilon})$. Therefore, we shall investigate the smoothness of v_ε . From Lemma 2.1 and Proposition 2.2 in [11] we have the decomposition of solution of (20),

$$v_\varepsilon = v_{r\varepsilon} + c_\varepsilon \overline{S}_\varepsilon,$$

where $v_{r\varepsilon} \in H^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega_\varepsilon})$, $\overline{S}_\varepsilon \in H^1(\Omega_\varepsilon) \setminus H^2(\Omega_\varepsilon)$, $c_\varepsilon = -2a \int_{\Gamma_{1,\varepsilon}} \overline{S}_\varepsilon - 2b \int_{\Gamma_{2,\varepsilon}} \overline{S}_\varepsilon$.

On the other hand, using a standard argument (proof of Lemma 2.1 [11]), we get

$$w_\varepsilon = w_{r\varepsilon} + \tilde{c}_\varepsilon \overline{S}_\varepsilon,$$

where $w_{r\varepsilon} \in H^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega_\varepsilon})$ and $\tilde{c}_\varepsilon = \int_{\Omega_\varepsilon} \overline{S}_\varepsilon f_\varepsilon$. Thus,

$$\underbrace{u_\varepsilon}_{\in H^2(\Omega_\varepsilon)} = v_\varepsilon + w_\varepsilon = \underbrace{(v_{r\varepsilon} + w_{r\varepsilon})}_{\in H^2(\Omega_\varepsilon)} + (c_\varepsilon + \tilde{c}_\varepsilon) \underbrace{\overline{S}_\varepsilon}_{\notin H^2(\Omega_\varepsilon)}.$$

Hence, $c_\varepsilon = -\tilde{c}_\varepsilon$ and we obtain

$$\int_{\Omega_\varepsilon} \overline{S}_\varepsilon f_\varepsilon = 2a \int_{\Gamma_{1,\varepsilon}} \overline{S}_\varepsilon + 2b \int_{\Gamma_{2,\varepsilon}} \overline{S}_\varepsilon. \quad (21)$$

We would like to show that v_ε is not in $C^1(\overline{\Omega})$ for small ε . For this purpose, it is enough to show that the left hand side of (21) is not zero. First, we notice that using Lemma 2.1, we conclude that $\varepsilon \mapsto \int_{\Omega_\varepsilon} \overline{S}_\varepsilon f_\varepsilon$ is continuous at 0.

Thus, the proof will be finished, if

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \overline{\overline{S}}_\varepsilon f_\varepsilon \neq 0. \quad (22)$$

Using Lemma 2.1 again we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \overline{\overline{S}}_\varepsilon f_\varepsilon dx dy &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon \cap \{|x| < \varepsilon\}} \overline{\overline{S}}_\varepsilon u_{yy}(0, y) dx dy \\ &= 2 \int_{r_2}^{\lambda_0 r_2} \overline{\overline{S}}(0, y) u_{yy}(0, y) dy = -2 \int_{r_2}^{\lambda_0 r_2} \overline{\overline{S}}(0, y) u_{xx}(0, y) dy. \end{aligned}$$

By Theorem 2.1 we deduce $\overline{\overline{S}}(0, y) < 0$ for $y \in (r_2, \lambda_0 r_2)$. Finally, in the second part of the proof of Theorem 3.1 we deduce that u_x is negative in $\Omega_+ = \Omega \cap \{x > 0\}$ and by symmetry $u_x = 0$ on $\Omega \cap \{x = 0\}$. Hence the function $u_x|_{\Omega_+}$ has its maximum on $\Omega \cap \{x = 0\}$. Therefore from Hopf Lemma we deduce that $u_{xx}(0, y) < 0$ for $y \in (r_2, \lambda_0 r_2)$, hence

$$-2 \int_{r_2}^{\lambda_0 r_2} \overline{\overline{S}}(0, y) u_{xx}(0, y) dy < 0,$$

and the proof is finished. \square

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